Optimal Queue Design

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Introduction

- Much of goods and services are allocated through non-market means, or "rationed," by far most commonly through "waiting in line."
- But waiting is costly and painful:
  - 6 months of life waiting in line for things (e.g., schools, hospitals, bookstores, libraries, banks, post office, petrol pumps, theatres...)
  - 43 days on hold with call centers (Brown et al. 2005)
- Challenge: How to efficiently provide incentives for waiting in line?
- Queue designer has several instruments at disposal (imagine call centers): entry and exit, queueing rule, the information policy.
- Existing queueing theory, in particular, *rational queueing (e.g., Hassin (2016))*, treats this issue, but in limited scopes both in terms of the design variables and agents' incentives.

What We Do

- We take a Myerson-style mechanism design approach to the design of the optimal queueing system,
- allowing all three aspects of queue design— entry/exit, queueing rule, and the information policy—chosen optimally
- while taking into consideration incentives for agents to join a queue and to stay in the queue, whenever necessary.
- We show: Under a mild condition, the optimal policy is implemented by **first-come-first-served** with the policy of **no information** given to the agents about the queue.

A queueing model with general Markov process

- Continuous time.
- **Primitive process:** At each instant,
  - an agent arrives (randomly) at a Poisson rate $\lambda_k > 0$
  - and is served (randomly) at a Poisson rate $\mu_k > 0$ (= average service time is $1/\mu_k$),

  when there are k agents in the queue.
- Dependence on k allows for extra generality: See next slides.
- Assume **Regularity:**

  (i) μ_k is nondecreasing and concave in k

  (ii) $\lambda_{k+1} - \lambda_k \leq \mu_{k+1} - \mu_k$, $\forall k$
- A mild assumption satisfied in virtually all realistic environments.

Examples:

- M/M/1: λ_k, μ_k do not depend on k
- M/M/c: λ_k does not depend on k and $\mu_k = \min\{k, c\}\mu$,
- Dynamic matching with stochastic compatibility
  - effective arrival rate $=$ arrival rate $\times$ prob of not compatible with anybody in the queue (depends on k)
  - effective exit rate $=$ arrival rate $\times$ prob of somebody in the queue being compatible (depends on k)

Preferences

Standard queueing model: homogeneous preferences with linear waiting costs.

- **Agents' payoffs:**
$$U(t) = V - C \cdot t,$$
  where t the time spent in the system.
  - $V > 0$: net surplus from service
  - $C > 0$: per-period cost of waiting
  - zero outside option.

- **The firm** receives $R > 0$ from each agent served

- **Designer's objective.** Weighted sum of firm's and agents' payoffs.

Queueing Mechanism

- Entry rule: $x = (x_k)$, where x_k is prob of recommending entry in a queue of length k

  "Please hold; somebody will be with you" or "... please come back some other time; good bye."

- Exit rule: $y = (y_{k,\ell})$, where $y_{k,\ell}$ is the rate of removal when queue length is k and position is ℓ; we also allow for a "lumpy" exit upon a new entry (omitted here)

  "We are experiencing unusual call volume, please come back later"

Queueing Mechanism—Continued

- Queueing rule: $q = (q_{k,\ell})$ where $q_{k,\ell}$ the service rate when queue length is k and position is ℓ, such that
  - Feasibility: For any set $S \subset \{1, ..., k\}$ of $|S| = m$ agents:
    $$\sum_{j \in S} q_{k,j} \leq \mu_m$$
  - Work-conservation:
    $$\sum_{\ell=1}^{k} q_{k,\ell} = \mu_k$$
- Examples:
  - First-Come First-Served (FCFS): $q_{k,1} = \mu_1, q_{k,2} = \mu_2 - \mu_1, ...,$ $q_{k,\ell} = \mu_\ell - \mu_{\ell-1}$. (M/M/1, $q_{k,\ell} = \mu$ if $\ell = 1$ and 0 o/wise)
  - Last-Come First-Served (LCFS): $q_{k,k} = \mu_1, q_{k,k-1} = \mu_2 - \mu_1, ...,$ $q_{k,\ell} = \mu_{k-\ell+1} - \mu_{k-\ell}$ (M/M/1, $q_{k,\ell} = \mu$ if $\ell = k$ and 0 o/wise)
  - Service-In-Random-Order (SIRO): $q_{k,\ell} = \mu_k / k$

Queueing Mechanism—Continued

- Information rule: $I = (I_t)$, where I_t specifies the information an agent gets about the state—i.e., the queue length k and her position ℓ—for each time $t \geq 0$ spent on the queue.

- Examples:
  - Full information
  - No information (beyond recommendations)

Overview

- The entry/exit rules (x, y), together with (λ, μ), induces a Markov chain on the queue length k with an invariant distribution $p = (p_k) \in \Delta(\mathbb{Z}_+)$.

- We focus on the problem at steady state, or invariant distribution:

"Maximize designer objective (at the invariant dist)

subject to: Agents have incentives to join and stay whenever needed."

- Why IC? Agents can be denied entry or removed without consent, but they cannot be coerced to join the queue or staying in it against their will.

Related Literature

- Queueing Design with fixed information rule:
  - Naor (1969), Hassin (1985), Su and Zenios (2004): Excessive incentives for queueing under FCFS, corrected by LCFS
  - Leshno (2019): Insufficient incentives for queueing under FCFS, corrected by SIRO or LIEW
  - Bloch and Cantala (2017), Margaria (2020),...
  - Ashlagi, Faidra, and Nikzad (2020)
- Information Design with fixed queueing rules:
  - Hassin and Koshman (2017), Lingenbrink and Iyer (2019), Anunrojwong, Iyer, and Manshadi (2020)
- Current work distinguished by:
  - the generality of the primitive Markov process and designer objective
  - the comprehensiveness of mechanism design approach
  - the consideration of dynamic incentive issues

Designer's problem

Designer chooses (x, y, q, I) to solve:

$[P]$ Maximize designer objective at p,

subject to

(B) p is an invariant distr given by (x, y)

and

(IC) incentives to join or stay when recommended

Designer's problem

Designer chooses (x, y, q, I) to solve:

$$[P] \qquad \text{Maximize } (1-\alpha)\sum_{k=1}^{\infty} p_k \mu_k R + \alpha \sum_{k=1}^{\infty} p_k(\mu_k V - kC),$$

subject to

$$(B) \qquad \lambda_k x_k p_k = (\mu_{k+1} + \sum_{\ell} y_{k+1,\ell}) p_{k+1},\ \forall k$$

and

(IC) Incentive constraints for every signal at each time t

Remark: Difficult to solve.

A relaxed LP problem

The designer chooses (only!) p

$$[P'] \qquad \text{Maximize } (1-\alpha)\sum_{k=1}^{\infty} p_k \mu_k R + \alpha \sum_{k=1}^{\infty} p_k(\mu_k V - kC),$$

subject to relaxation of balance equation:

$$(B') \qquad \lambda_k p_k - \mu_{k+1} p_{k+1} \geq 0$$

subject to relaxed incentive compatibility:

$$(IR) \qquad \sum_{k=1}^{\infty} p_k(\mu_k V - kC) \geq 0.$$

Remark: (IR) equivalent to "agents having incentives to join under no information."

Optimality of Cutoff Policy

Theorem

If μ is regular, then an optimal solution of relaxed program $[P']$ is a cutoff policy, meaning there exists $K^* \geq 0$ such that agents are allowed to queue up to K^*.

Note: Random rationing possible at $K^* - 1$.

Implication: No need for removing agents.

Optimality of FCFS with no information

Theorem

Assume the primitive process is regular. **FCFS + no information** *(i.e., beyond that inferred by recommendation) is optimal.*

- Can implement the cutoff policy that solves the relaxed program $[P']$ with FCFS + No information.
- Proof:
  1. Incentives to join the queue: Holds since (IR) is satisfied at the solution.
  2. Incentives to stay in the queue until served: non-trivial.

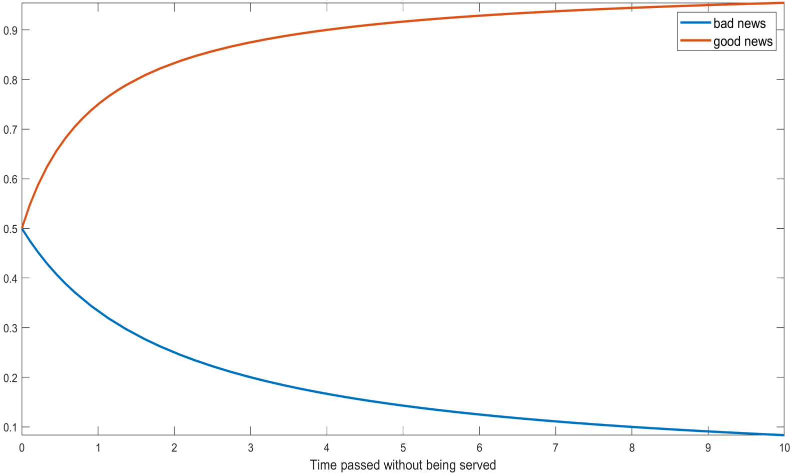
     We show: Under regularity, beliefs about queue position improve over time $\Rightarrow$ Residual waiting time falls.

Intuition

- Question: Is "time spent in the queue" good news or bad news?
  - Good news: *conditional on the initial queue length*, under FCFS, position in queue can only improve
  - Bad news: "the initial queue length may have been longer" $\Rightarrow$ pessimistic updating

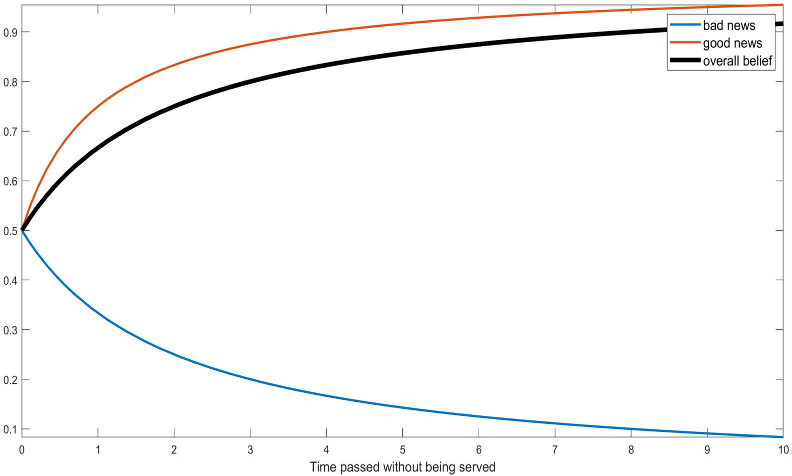
We show that, given the regularity of the primitive process, good news dominates bad news.

Belief about position $\ell = 1$



M/M/1 with $K^* = 2$; $\lambda = \mu = 1$.

Belief about position $\ell = 1$



M/M/1 with $K^* = 2$; $\lambda = \mu = 1$.

Evolution of beliefs under FCFS with no information

- γ_ℓ^t = belief that position is ℓ after spending time $t \geq 0$ on the queue.
- Consider **likelihood ratios**: $r_\ell^t \triangleq \frac{\gamma_\ell^t}{\gamma_{\ell-1}^t}$, for all $\ell = 2, \ldots, K^*$.
- Suffices to show: the likelihood ratios $(r_\ell^t)_\ell$ fall in t.

  $\Rightarrow$ Beliefs about queue position improve over time

  $\Rightarrow$ Residual waiting time falls.

Do the likelihood ratios fall?

- How does (r_ℓ^t) evolve?

$$r_\ell^{t+dt} = \frac{\gamma_\ell^{t+dt}}{\gamma_{\ell-1}^{t+dt}} = \frac{(1-\mu_\ell dt)\gamma_\ell^t + (\mu_\ell dt)\gamma_{\ell+1}^t}{(1-\mu_{\ell-1} dt)\gamma_{\ell-1}^t + (\mu_{\ell-1} dt)\gamma_\ell^t} + o(dt).$$

$\Rightarrow$ System of ODEs of the likelihood ratios:

$$\dot{r}_\ell^t = r_\ell^t \left(-(\mu_\ell - \mu_{\ell-1}) + (\mu_\ell r_{\ell+1}^t - \mu_{\ell-1} r_\ell^t)\right)$$

- Generally ambiguous. The "initial beliefs" matter!

Evolution of beliefs under FCFS with no information

- The likelihood ratios at $t = 0$ given by the invariant distr. (cf. PASTA): $\forall \ell = 2, ..., K^*$,

$$
\begin{aligned}
\dot{r}_\ell^0 &= r_\ell^0 \left(-(\mu_\ell - \mu_{\ell-1}) + (\mu_\ell r_{\ell+1}^0 - \mu_{\ell-1} r_\ell^0) \right) \\
&= r_\ell^0 \left(-(\mu_\ell - \mu_{\ell-1}) + (\mu_\ell \frac{\lambda_\ell}{\mu_\ell} - \mu_{\ell-1} \frac{\lambda_{\ell-1}}{\mu_{\ell-1}}) \right) \\
&= r_\ell^0 \left(-(\mu_\ell - \mu_{\ell-1}) + (\lambda_\ell - \lambda_{\ell-1}) \right) \leq 0.
\end{aligned}
$$

- The system of ODEs is "cooperative":

$$
\dot{r}^0 \leq 0 \Rightarrow \dot{r}^t \leq 0 \text{ for all } t
$$

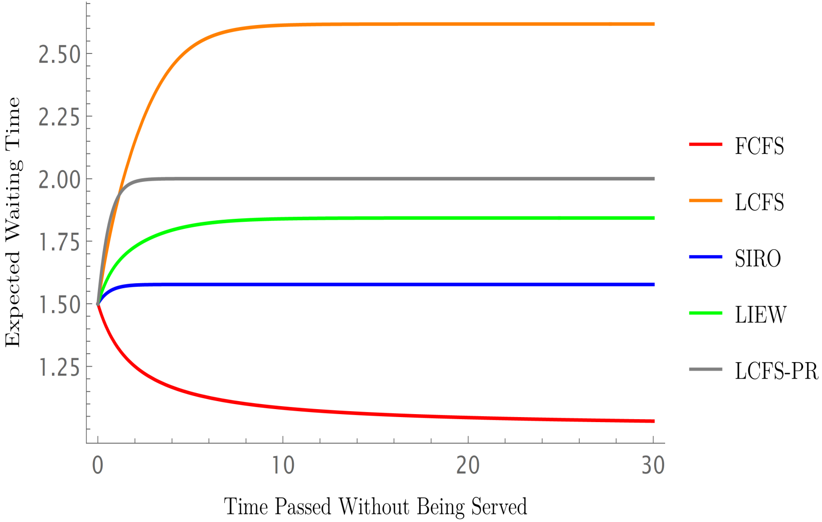
Necessity of FCFS for Optimality

In principle, other queueing rules or information rules may work under some environments. But FCFS with no information is uniquely optimal in a *maximal domain sense*

Theorem

For any queueing rule differing from FCFS, there exists a queueing environment (λ, μ, V, C) *under which the rule can't implement the optimal policy regardless of the information policy.*

Residual waiting time under alternative queueing rules.



M/M/1 with $K^* = 2$; $\lambda = \mu = 1$.

Concluding Thoughts

- Without info design, FCFS is typically suboptimal, and optimal policy is unknown and difficult to find.
- With information design, FCFS is (uniquely) optimal
- Of course, there may be unmodeled benefits of providing information on queue position or expected waiting times
  - intrinsic value of transparency
  - ambiguity aversion...
- We have identified a novel role for queueing disciplines in regulating agents' beliefs, and their dynamic incentives
- Revealed a hitherto-unrecognized virtue of FCFS in this regard.

Thank You!

References

Anunrojwong, J., K. Iyer, and V. Manshadi (2020): "Information Design for Congested Social Services: Optimal Need-Based Persuasion," *EC '20: Proceedings of the 21st ACM Conference on Economics and Computation*, pp. 349–350.

Ashlagi, I., M. Faidra, and A. Nikzad (2020): "Optimal Dynamic Allocation: Simplicity through Information Design," Discussion paper, Stanford.

Bloch, F., and D. Cantala (2017): "Dynamic assignment of objects to queuing agents," *American Economic Journal: Microeconomics*, 9, 88–122.

Hassin, R. (1985): "On the optimality of first come last served queues," *Econometrica*, 53, 201–202.

Hassin, R. (2016): *Rational Queueing*. CRC Press.

Hassin, R., and A. Koshman (2017): "Profit maximization in the M/M/1 queue," *Operations Research Letters*, 45, 436–441.

Leshno, J. (2019): "Dynamic matching in overloaded waiting lists," Discussion paper, SSRN Working Paper 2967011.

Lingenbrink, D., and K. Iyer (2019): "Optimal signaling mechanisms in unobservable queues," *Operations Research*, 67, 1397–1416.

Margaria, C. (2020): "Queueing to learn," Discussion paper, Boston University.